The Cosmic Quartet
Cosmological Parameters of a Smoothed Inhomogeneous Spacetime

Thomas Buchert[a,1] and Mauro Carfora[b,2]

[a] *Theoretische Physik, Ludwig–Maximilians–Universität, Theresienstr. 37, D–80333 München, Germany and Department of Physics and Research Center for the Early Universe (RESCEU) School of Science, The University of Tokyo, Tokyo 113–0033, Japan*

[b] *Dipartimento di Fisica Nucleare e Teorica, Università degli Studi di Pavia and Istituto Nazionale di Fisica Nucleare, Sezione di Pavia, via A. Bassi 6, I–27100 Pavia, Italy*

Abstract

We discuss the relation between 'bare' cosmological parameters as the true spatial average characteristics that determine the cosmological model, and the parameters interpreted by observers with a "Friedmannian bias", i.e., within a homogeneous space geometry. We may say that the latter are 'dressed' by the smoothed–out geometrical inhomogeneities of the surveyed spatial region. We identify two effects that quantify the difference between 'bare' and 'dressed' parameters: 'curvature backreaction' and 'volume effect'. An estimate of the latter is given in terms of a simple geometrical example.

1 Introduction

Contemporary cosmology is heading, as is the generally held believe, towards a theoretical home–stretch marked by the determination of three cosmological parameters that characterize the standard model of cosmology and with it the whole global history of our Universe. This standard model looks back on a long period of challenges that were mastered by its apparent robustness. Essentially, it is based on solutions of Einstein's equations for a homogeneous and isotropic distribution of matter in the Universe. So, it is indeed a simple model that was iconed by Neta Bahcall et al. [1] in terms of a 'cosmic triangle'. The final phase of combat, in which we are in at present and as a large body of cosmologists would put it, consists of a phase of high–precision cosmology which, through a plethora of orthogonal observations, will soon exclaim highly accurate numbers for those three cosmological parameters. But, should we really believe this enormously simplified pledge of the standard model en–chaining predictions throughout all space and time?

A more level–headed approach immediately reminds us of the foundations of the standard model: it ignores structure in the Universe. Of course, nobody is as naive as ignoring the existence of structure, which in the recent past was celebrated by a number of observational successes: the discovery of large–scale structure, essentially blowing up the picture of a homogeneous Universe and replacing it by a strongly proncounced honeycomb structure of the distribution of galaxies. Those structures are extending as deep as hundred millions of lightyears and are still continuing at the borders of our maps of the Universe such as that currently drawn by the Sloan Digital Sky Survey group. The "excuse" is a conjecture: to maintain the standard model we have to assume that *on average* the cosmic evolution follows the Friedmannian equations whose solutions are fixed by the cosmic triangle.

Guided by the idea that a universe model should remain simple, while respecting the inhomogeneities that are present in the matter distribution as well as in the geometry of spacetime, we have worked out a cosmology that can still be characterized by 'cosmological parameters', but there are essentially three important colors that have to be added to the standard picture: firstly, replacing the parameters of the homogeneous universe model by corresponding spatial averages entails a scale–dependence, i.e., the values of the parameters must depend on the spatial scale over which we average the distribution. Secondly, in addition to the standard model parameters, there is a fourth player which encodes the

[1] E-mail:buchert@theorie.physik.uni-muenchen.de
[2] E-mail:mauro.carfora@pv.infn.it

fluctuations of matter inhomogeneities [3]. If averaged on sufficiently large scales this new parameter may be a small quantity, but it adds a substantial qualitative impact: cosmological parameters are now allowed to communicate, they evolve freely governed by the evolution of structure and they are not fixed each individually by initial conditions as in the standard model [6]; we may say that the 'cosmic triangle' is transformed into a 'cosmic terzet'. Thirdly, and this turns out to be a more deeply–rooted insight: cosmological parameters that we would measure and interprete with a "Friedmannian bias" (i.e. in a homogeneous spacetime geometry) are 'dressed' [8], i.e., they acquire correction factors stemming from the geometrical inhomogeneities.

Understanding this latter fact took a long and rather challenging route of employing a number of results that were obtained by mathematicians who work on Riemannian geometry [7], reinforcing the Ricci flow as a natural candidate for the smoothing of geometry [12],[10]. We shall demonstrate in this note that 'the emperor's new clothes' can be seen, and the effect of 'dressing' can be calculated. Possibly, the values interpreted with a Friedmannian bias are appreciably different from the true averages that govern the dynamics of spacetime. Further work must be devoted to quantifying this effect: could we eventually dismiss the cosmological constant, which now seems to be an unavoidable thigh of the cosmic triangle? It will be exciting to follow this new route of understanding the influence of structure of the fluctuating geometry of spacetime itself on the evolution of our Universe.

2 Bare and Dressed Cosmological Parameters

To begin with let us concentrate on the density parameter which measures the actual matter content in a spatial slice of the Universe on some spatial scale of observation. We assume that we are given a suitable split into space and time, i.e., a foliation of spacetime. Since the determination of the *total mass on a regional domain* in relativistic cosmology is a subject of controversy (see, e.g., [2]), we only make use of the well–defined concepts of a local energy density ϱ (here, we only refer to the restmass density for the matter model 'dust'), and its spatial Riemannian volume average evaluated on a compact domain, e.g., a geodesic ball $\mathcal{B}_0$, $\langle\varrho\rangle_{\mathcal{B}_0} := \int_{\mathcal{B}_0}\varrho d\mu_g / V_{\mathcal{B}_0}$ with the Riemannian volume element $d\mu_g$ associated with the 3–metric of the hypersurface, and the ball's volume $V_{\mathcal{B}_0} = \int_{\mathcal{B}_0} d\mu_g$. Using this averager for all scalar variables in a $3+1$ ADM setting, we can investigate regional averages of all relevant spatial variables such as the density and, e.g., the expansion rate $\langle\theta\rangle_{\mathcal{B}_0} =: 3H_{\mathcal{B}_0}$, defining an effective "Hubble–constant" on the domain of averaging. Hence, the cosmological density parameter that determines the true ('bare') source of the effective dynamics can be unambiguously defined.

The averaged Hamiltonian constraint may be cast into a relation among a set of *regional cosmological parameters* as the following scale–dependent functionals [3]:

$$\Omega^M_{\mathcal{B}_0} := \frac{8\pi G M_{\mathcal{B}_0}}{3V_{\mathcal{B}_0}H^2_{\mathcal{B}_0}} \;\;;\;\; \Omega^\Lambda_{\mathcal{B}_0} := \frac{\Lambda}{3H^2_{\mathcal{B}_0}} \;\;;\;\; \Omega^R_{\mathcal{B}_0} := -\frac{\langle\mathcal{R}\rangle_{\mathcal{B}_0}}{6H^2_{\mathcal{B}_0}} \;\;;\;\; \Omega^{\mathcal{Q}^K}_{\mathcal{B}_0} := -\frac{\mathcal{Q}^K_{\mathcal{B}_0}}{6H^2_{\mathcal{B}_0}} \;\;,$$

$$\text{obeying by construction the Hamiltonian constraint} \quad \Omega^M_{\mathcal{B}_0} + \Omega^\Lambda_{\mathcal{B}_0} + \Omega^R_{\mathcal{B}_0} + \Omega^{\mathcal{Q}^K}_{\mathcal{B}_0} = 1 \;\;, \tag{1}$$

with the ball's material mass content $M_{\mathcal{B}_0}$, the scalar curvature $\mathcal{R}$, and the cosmological constant Λ. In contrast to the standard FLRW cosmological parameters there are four players. In the FLRW case there is by definition no fluctuating source (condensed into the "kinematical backreaction" $\mathcal{Q}^K_{\mathcal{B}_0}$ that is composed of positive–definite expansion and shear fluctuation terms (see [3]). Hence, the *effective cosmology* as the spatially averaged model can be determined by a scale–dependent and regional 'cosmic quartet' [4] rather than by a global 'cosmic triangle' [1].

Eq. (1) forms the basis of a discussion of cosmological parameters as they determine the theoretical model. They may not be, however, directly accessible to observations. Unlike in Newtonian cosmology, where the corresponding averages have a similar form [5], it is not straightforward to compare the above relativistic average model parameters to observational data. The reason is that the volume–averages contain information on the actually present *geometrical inhomogeneities* within the averaging domain. In contrast, the "observer's Universe" is described in terms of a Euclidean or constant curvature model. Consequently, the common interpretation of observations within the set of the standard model parameters, if extended by the backreaction parameter or not, neglects the geometrical inhomogeneities that (through the Riemannian volume–average) are hidden in the average characteristics of the theoretical cosmology.

We have suggested an answer to this problem in [7] (see also [9] for a preliminary attempt). Let us highlight some results.

According to [7], the picture discussed above strictly depends on the ratio between two density profiles defined on the averaging domain: one is naturally associated with the actual matter content of the gravitational sources, whereas the other is the mass density corresponding to the matter content in the given region, but now thought of as averaged over a geometrically smoothed–out domain $\overline{\mathcal{B}}$ with homogeneous geometry:

$$\langle \varrho \rangle_{\mathcal{B}_0} = M_{\mathcal{B}_0}/V_{\mathcal{B}_0} \quad ; \quad \langle \varrho \rangle_{\overline{\mathcal{B}}} = M_{\overline{\mathcal{B}}}/V_{\overline{\mathcal{B}}} \ . \tag{2}$$

We have implemented a regional smoothing procedure for the geometry of the hypersurface under the assumption of preservation of the ball's material content. We can infer already from (2) that the average density measured with a "Friedmannian bias" is dressed by a *volume effect* due to the difference between the volume of a smoothed region and the actual volume of the bumpy region.

A further result that explicitly involves the geometrical smoothing flows (e.g., the Ricci flow for the metric) is furnished by a relation between the constant regional curvature in the smoothed model (e.g., a FLRW domain) and the actual regional average curvature in the theoretical cosmology:

$$\overline{\mathcal{R}}_{\overline{\mathcal{B}}} = \langle \mathcal{R} \rangle_{\mathcal{B}_0} (V_{\overline{\mathcal{B}}}/V_{\mathcal{B}_0})^{-2/3} - \mathcal{Q}^R_{\mathcal{B}_0} \ , \tag{3}$$

where we have introduced a novel measure for the "backreaction" of geometrical inhomogeneities capturing the deviations from the standard FLRW space section, the *regional curvature backreaction*:

$$\mathcal{Q}^R_{\mathcal{B}_0} := \int_0^\infty d\beta \ \frac{V_{\mathcal{B}_\beta}(\beta)}{V_{\overline{\mathcal{B}}}} \ \left[\frac{1}{3} \langle \big(\mathcal{R}(\beta) - \langle \mathcal{R}(\beta) \rangle_{\mathcal{B}_\beta} \big)^2 \rangle_{\mathcal{B}_\beta} - 2 \langle \tilde{\mathcal{R}}^{ab}(\beta) \tilde{\mathcal{R}}_{ab}(\beta) \rangle_{\mathcal{B}_\beta} \right] \ , \tag{4}$$

with $\tilde{\mathcal{R}}_{ab} := \mathcal{R}_{ab} - \frac{1}{3} g_{ab} \mathcal{R}$ being the trace–free part of the Ricci tensor $\mathcal{R}_{ab}$ in the hypersurface. $\mathcal{Q}^R_{\mathcal{B}_0}$, built from scalar invariants of the intrinsic curvature, appears to have a similar form as the "kinematical backreaction" term (that was built from invariants of the extrinsic curvature). It features two positive–definite parts, the *scalar curvature amplitude fluctuations* and *fluctuations in metrical anisotropies*. Depending on which part dominates we obtain an under– or overestimate of the actual averaged scalar curvature, respectively. β parametrizes integral curves of the smoothing flow for the metric, so that the expression above indeed refers to the explicit form of this flow. Notwithstanding, this term may be estimated by the actual curvature fluctuations, since the Ricci flow acts in a controllable way such that the maxima of the curvature inhomogeneities are monotonically decreasing during the deformation.

From Eq. (3) we can understand the physical content of geometrical averaging. It makes transparent that, in the smoothed model, the averaged scalar curvature is 'dressed' both by the *volume effect* mentioned above, and by the *curvature backreaction effect* itself. The volume effect is expected precisely in the form occurring in (3), if we think of comparing two regions of distinct volumes, but with the same matter content, in a constant curvature space. Whereas the backreaction term encodes the deviation of the averaged scalar curvature from a constant curvature model, e.g., a FLRW space section.

Correspondingly, an observer with a "Friedmannian bias" would interprete his measurements in terms of the 'dressed' cosmological parameters:

$$\overline{\Omega}^M_{\overline{\mathcal{B}}} := \frac{8\pi G M_{\overline{\mathcal{B}}}}{3 V_{\overline{\mathcal{B}}} \overline{H}^2_{\overline{\mathcal{B}}}} \ ; \ \overline{\Omega}^\Lambda_{\overline{\mathcal{B}}} := \frac{\Lambda}{3\overline{H}^2_{\overline{\mathcal{B}}}} \ ; \ \overline{\Omega}^R_{\overline{\mathcal{B}}} := -\frac{\overline{\mathcal{R}}_{\overline{\mathcal{B}}}}{6\overline{H}^2_{\overline{\mathcal{B}}}} \ ; \ \overline{\Omega}^{\mathcal{Q}^K}_{\overline{\mathcal{B}}} := -\frac{\overline{\mathcal{Q}}^K_{\overline{\mathcal{B}}}}{6\overline{H}^2_{\overline{\mathcal{B}}}} \ , \ \text{obeying} \quad \overline{\Omega}^M_{\overline{\mathcal{B}}} + \overline{\Omega}^\Lambda_{\overline{\mathcal{B}}} + \overline{\Omega}^R_{\overline{\mathcal{B}}} + \overline{\Omega}^{\mathcal{Q}^K}_{\overline{\mathcal{B}}} = 1 \ . \tag{5}$$

The latter equation follows from our assumption that the smoothing procedure requires to respect the Hamiltonian constraint of Einstein's equations. (For a more detailed discussion see [8].)

3 The Volume Effect: a Geometrical Example

Comparing and using the relations (1) and (5), we conclude that the problem has been condensed into a recipee–type rule of applying results that were obtained on theoretically quite involved grounds. Notwithstanding, the formulation appears simple and therefore immediately suggests questions of practical importance. Concrete numbers are needed in order to persuade workers in observational cosmology of

the actual quantitative importance of the outlined effects. However, to give concrete numbers needs a substantial effort beyond the currently available methods. One of the obstacles to provide immediate estimates is the dependence of the proposed effects on generic dynamical models, which are not well–developed in relativistic cosmology. To employ (non–generic) spherically symmetric models is one way to go, and we are currently conducting research in this direction.

Hellaby [13] used spherically symmetric models of clusters and voids to obtain the error made by estimating average characteristics in comparison with parameters of a corresponding FLRW cosmology. The notion 'corresponding' is defined in the sense of *volume matching* proposed earlier by Ellis and Stoeger [11]. To compare Hellaby's approach with our framework is difficult in detail. The volume fraction which would have to be estimated in our framework is 1 by definition, the 'curvature backreaction' being the only geometrical effect. But, since Hellaby compares scalar average characteristics that involve the averaged curvatures and the averaged densities on a given scale of interest by keeping the volumes the same, the calculated effect still can be used as an indication for the deviations of the averages due to geometrical differences of the two models. It must be emphasized, however, that a detailed comparison has to follow from an explicit investigation of our effects, and therefore Hellaby's quoted numbers (amounting to 10–30% for typical clusters and voids) can at best be used as an indication. Moreoever, as Hellaby pointed out himself, the effect depends on details of the dynamical model which, in the cases he investigated, are still simple solutions that can be done analytically. Also, as he mentioned, the effect would strongly depend on the topology of the domain which he chooses to be the simplest.

A complementary aspect of the problem is that we need to analyze realistic models of hypersurfaces in order to learn about the global effect on very large scales; note that, even if we have realistically modelled clusters and voids, we would still need to arrange those patterns according to knowledge of the distribution of large–scale structure. In this sense the effect calculated by Hellaby (loc.cit.) is plausible, but it will not provide the answer on cosmological scales. Let us illustrate this comment: researches employing rough–surface models to realistically model e.g. the moon's surface for the purpose of better estimates on reflection intensity of light from the surface, would work with factors of 40 for the fraction of the moon's modelled inhomogeneous surface to the surface of a sphere (Karl Vogler, priv. comm.). For cosmological volumes the situation is more subtle, since different patterns may increase, but they may also decrease the volume of the spatial domain. Concerning this problem we are currently investigating a generic relativistic model based on a relativistic generalization of Zel'dovich's approximation, which is a well–known approximate tool in Newtonian cosmology to model generic structure formation [6].

For the sake of illustration of the effect of the volume scaling on the mass density, let us consider the following explicit example. Its construction requires the following steps:

(i) Take a large Euclidean manifold, say a flat 3–torus T^3 of *typical* size $R \times R \times R$.

(ii) Remove from such a manifold, k Euclidean balls $B_E(\frac{\pi}{2}r)$ of radius $\frac{\pi}{2}r$ with $\frac{r}{R}$ small, (the $\frac{\pi}{2}$ is for computational convenience).

(iii) Glue, in place of the removed Euclidean balls $\{B_E(\frac{\pi}{2}r)\}$, a corresponding number k of 3–spherical balls $\{B_{S^3}(\frac{\pi}{2}r)\}$, (*i.e.*, balls cut out of the surface of a 3–sphere S^3_r, if the 3–sphere has radius r then the ball $B_{S^3}(\frac{\pi}{2}r)$ is one hemisphere of S^3_r).

(iv) Note that such a glueing requires a *small* amount of negative curvature along the boundary 2–spheres of $B_E(\frac{\pi}{2}r)$ in order to have a smooth interface between the flat geometry of the torus T^3 and the positive curvature geometry of the 3–spherical balls $\{B_{S^3}(\frac{\pi}{2}r)\}$.

(v) Denote by $T^3_{\rm inh}$ the resulting curved torus, where the $B_{S^3}(\frac{\pi}{2}r)$'s correspond to the local geometric inhomogeneity over a region of radius $\frac{\pi}{2}r$ in a space section of the *observed* universe, (a sort of naive Swiss–cheese model).

Under such assumptions (typically, if r/R is suitably small) it is reasonable to conjecture that the geometry of $T^3_{\rm inh}$ will Ricci flow towards the geometry of a flat torus $\overline{T^3}$ (with $\mathrm{Vol}(\overline{T^3}) = \mathrm{Vol}(T^3_{\rm inh})$), and we can estimate the relative density fraction $\frac{\langle\varrho\rangle_{B_{\rm inh}}}{\langle\varrho\rangle_{B_E}}$ in such a setting. Note that the generic region $B_{S^3}(\frac{\pi}{2}r)$ supporting the local inhomogeneity in $T^3_{\rm inh}$ corresponds to a Euclidean ball $B_E(\frac{\pi}{2}r)$ in the flat smoothed–out $\overline{T^3}$.

Under the local mass preservation constraint, the actual $\langle \varrho \rangle_{B_{S^3}}$ and the smoothed–out $\langle \varrho \rangle_{B_E}$ mass densities are related to each other by (2):

$$\frac{\langle \varrho \rangle_{B_{S^3}}}{\langle \varrho \rangle_{B_E}} = \frac{\mathrm{Vol}\left(B_E(\frac{\pi}{2}r)\right)}{\mathrm{Vol}\left(B_{S^3}(\frac{\pi}{2}r)\right)} . \tag{6}$$

Recall that

$$\mathrm{Vol}\left(B_{S^3}(\frac{\pi}{2}r)\right) = \mathrm{Vol}\left(S^2(r)\right) \int_0^{\frac{\pi}{2}r} \sin^2\left(\frac{t}{r}\right) dt = \pi^2 r^3 \ , \tag{7}$$

where $\mathrm{Vol}\left(S^2(r)\right)$ is the area of the 2–sphere of radius r. Since

$$\mathrm{Vol}\left(B_E(\frac{\pi}{2}r)\right) = \frac{4}{3}\pi \left(\frac{\pi}{2}r\right)^3 = \frac{\pi^4}{6}r^3 \ , \tag{8}$$

we get

$$\frac{\langle \varrho \rangle_{B_{S^3}}}{\langle \varrho \rangle_{B_E}} = \frac{\mathrm{Vol}\left(B_E(\frac{\pi}{2}r)\right)}{\mathrm{Vol}\left(B_{S^3}(\frac{\pi}{2}r)\right)} = \frac{\pi^2}{6} \simeq 1.6449 \ . \tag{9}$$

This shows that indeed one can have a significant mismatch between the actual average mass density and the smoothed–out mass density owing only to the volume effect (the curvature backreaction effect was not considered in this rather qualitative example).

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